Morphing Planar Graphs in Spherical Space

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Abstract. We consider the problem of intersection-free planar graph morphing, and in particular, a generalization from Euclidean space to spherical space. We show that under certain conditions, there exists a continuous and intersection-free morph between two sphere drawings of a maximally planar graph, where sphere drawings are the spherical equivalent of plane drawings: crossings-free geodesic-arc drawings. In addition to the existence proof, we describe a morphing algorithm along with its implementation.

1 Introduction

Morphing refers to the process of transforming one shape (the source) into another (the target). Morphing is widely used in computer graphics, animation, and modeling; see a survey by Gomes *et al* [9]. In planar graph morphing we would like to transform a given source graph to another pre-specified target graph. A smooth transformation of one graph into another can be useful when dealing with dynamic graphs and graphs that change through time where it is crucial to preserve the mental map of the user. The mental map preservation is often accomplished by minimizing the changes to the drawing and by creating smooth transitions between consecutive drawings.

In this paper we consider the problem of morphing between two drawings, D_s and D_t, of the same maximally planar graph $G = (V, E)$ on the sphere. The source drawing D_s and the target drawing D_t are sphere drawings (generalizations of Euclidean plane drawings to spherical space). The main objective is to find a continuous and intersection-free morph from D_s to D_t.

1.1 Previous Work

Morphing has been extensively studied in graphics, animation, modeling and computational geometry, e.g., morphing 2D images [3, 11], polygons and polylines [8, 15], 3D objects [12, 13] and free form curves [14].

Graph morphing, refers to the process of transforming a given graph G_1 into another graph G_2. Early work on this problem includes a result by Cairns in 1944 [4] who shows that if G_1 and G_2 are maximally planar graphs with the same embedding, then there exists a non-intersecting morph between them. Later, Thomassen [16] showed that if G_1 and G_2 are isomorphic convex planar graphs with the same outer face, then there exists a non-intersecting morph between them that preserves convexity. Erten *et al* show how to morph between drawings with straight-line segments, bends, and curves [6]. This algorithm makes use of compatible triangulations [2] and the convex representation of a graph via barycentric coordinates [7, 17].

As the sphere and the plane are topologically the same, it is natural to attempt to generalize the non-intersecting morph algorithm from Euclidean space to spherical space. Alfeld *et al* [1] and Gotsman *et al* [10] define analogues of barycentric coordinates on the sphere, for spherical Bernstein-Bézieri polynomials and for spherical mesh parameterization, respectively. However, barycentric coordinates are problematic on the sphere. One problem is that unlike on the Euclidean plane, three points on the sphere define two finite regions. A system of barycentric coordinates must distinguish between these two regions. A second problem arises from the non-linearity introduced by the sphere. The system of equations used to determine the drawing at any stage of the morph has non-unique solutions, and it is not easy to guarantee smoothness of the morph.

1.2 Our Results

Our approach to morphing spherical drawings focuses on affine transformations of the inscribed polytopes of the given spherical drawings. We apply rotations, translations, scaling and shearing to the inscribed polytope, while projecting its endpoints onto the surface of the sphere throughout the transformations. At an intermediate stage, we use the intersection-free morphing algorithm for plane drawings together with a gnomonic projection to/from the sphere. Our approach yields a continuous and intersection-free morph for sphere drawings of maximally planar graphs, provided that the source and target drawings have convex inscribed polytopes. Note that in general, the inscribed polytope of a sphere drawing is star-shaped but need not necessarily be convex.

2 Background

We begin with some mathematical background about sphere drawings and spherical projections. The concept of a straight line in Euclidean space generalizes to that of a *geodesic* in Riemannian spaces, where the geodesic between two points is defined as a continuously differentiable curve of minimal length between them. Thus, geodesics in Euclidean geometry are straight lines, and in spherical geometry they are arcs of great circles. The generalization of a crossings-free straight-line drawing of a planar graph in spherical space uses geodesics instead of straight-lines.

Definition 1. A *sphere embedding* of a graph is a clockwise order of the neighbors for each vertex in the graph. A drawing is a drawing of an embedding if neighbors of nodes in the drawing match the order in the embedding. Note that 3-connected planar graphs in general, and maximally graphs in particular, have a unique sphere embedding, up to reflection.

Definition 2. A *geodesic-arc sphere drawing* of a graph is the sphere analogue of a *straight-line drawing* of a graph. The drawing is determined entirely by a mapping of the vertices of the graph onto the sphere. An edge between two nodes is drawn as the geodesic arc between them. We assume that no two nodes are antipodal, as there is no unique geodesic arc between two antipodal points.

Definition 3. A *crossing-free, geodesic-arc sphere drawing* of a graph is a sphere drawing of the graph in which no two edges intersect, except at a node on which they are both incident. We refer to such drawings as *sphere drawings* for short. Note that sphere drawings are a generalization of straight-line plane drawings from Euclidean space to spherical space.

Definition 4. Given a sphere drawing D of a planar graph G, the *inscribed polytope* P of D is obtained by replacing the (geodesic) edges in the spherical drawing by straight-line segments. The inscribed polytope P is by definition simple and star-shaped, but not necessarily convex.

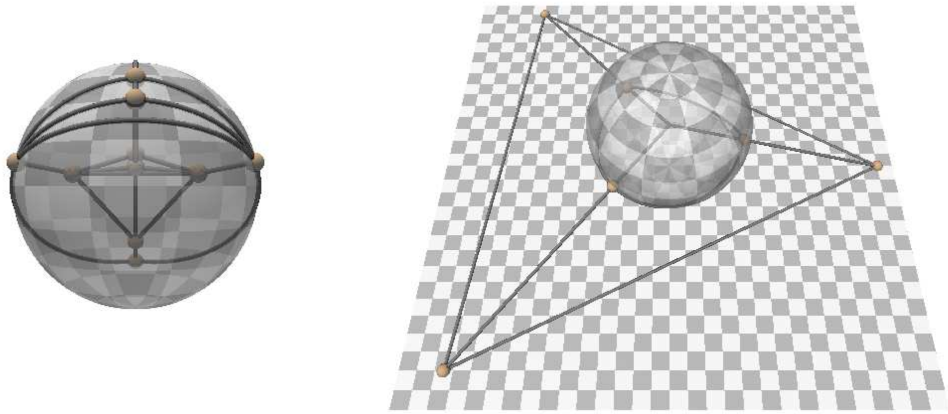
Definition 5. The *gnomonic projection* is a non-conformal map projection obtained by projecting a point on the surface of the sphere from the sphere's center to the point in a plane that is tangent to the south pole. Since this projection sends antipodal points to the same point in the plane, it can only be used to project one hemisphere at a time. In a gnomonic projection, geodesics are mapped to straight lines and vice versa [5].

3 Morphing between sphere drawings

The algorithm for morphing between two sphere drawings D_s and D_t of the same underlying graph G can be broken into several stages:

1. Choose an outer face f_0 of the underlying graph;
2. Morph the source sphere drawing D_s of G into D'_s, where D'_s is a sphere drawing of G such that the north pole is inside f_0 and the entire drawing is below the equator;
3. Morph the target sphere drawing D_t of G into D'_t, where D'_t is a sphere drawing of G such that the north pole is inside f_0 and the entire drawing is below the equator;
4. Project D'_s and D'_t using a gnomonic projection onto the plane tangent to the south pole to the drawings D''_s and D''_t;
5. Morph D''_s into D''_t using the graph morphing algorithm for plane drawings [6].

In practice, step 3 of the above algorithm is used in the reverse direction and altogether, the morphing sequence is: $D_s \rightarrow D'_s \rightarrow D''_s \rightarrow D''_t \rightarrow D'_t \rightarrow D_t$. By the definition of a gnomonic projection, since D'_s and D'_t are both strictly in the lower hemisphere, their projections D''_s and D''_t onto the plane tangent to the south pole are plane drawings. This implies the correctness of steps 4 and 5 and so, to argue the correctness of the overall approach, we must show that steps 2 and 3 of the algorithm above can be accomplished without introducing crossings in the morph.



(a) (b)

Fig. 1. (a) Projecting from a polytope that contains the origin to the surface of the sphere; (b) Gnomonic projection to and from the sphere.

3.1 Maintaining a Smooth and Crossings-Free Morph

Our approach to morphing sphere drawings uses a series of affine transformations to the inscribed polytope of the underlying graph (steps 2 and 3). We also rely on the barycentric morphing approach for plane drawings (steps 4 and 5). Thus, throughout the morph of our sphere drawing, we often track two positions for each vertex: the actual position of the vertex on the sphere in the sphere drawing, and the other, in some other construct. The other construct is either a 3D polytope, as in Fig. 1(a), or a plane drawing, as in Fig. 1(b). When transformations to the construct are applied, the positions of the vertices on the sphere change appropriately. A useful visualization for this approach is to imagine a spoke for each vertex, going from the origin of the sphere through both positions associated with that node. As one position changes, so does the other. For simplicity, assume the sphere is centered at $(0, 0, 0)$ with radius 1.

Theorem 1. *A strictly convex polytope containing the center of a sphere yields a sphere drawing of that polytope's skeletal graph when its vertices are normalized to lie on the sphere.*

Proof Sketch: First, note that the geodesic arc between two vertices on the sphere is the same as the projection of the straight line between those two vertices of the polytope. Suppose that the projection of the polytope onto the sphere has a crossing. Consider the point p on the sphere where two edges intersect. This point must be the projection of two different polytope edges onto the sphere. This implies that there exists a ray that starts at the center and intersects two separate edges of the polytope. Let p_1 and p_2 be the two points obtained from the intersection of each of these edges with the ray through the origin. Without loss of generality, let p_1 be the point that is further from the center. Then there exists a line segment from the center of the sphere to p_1 that passes through p_2. This contradicts the assumption that the polytope is strictly convex. Hence, the resulting sphere drawing must be crossing free. □

Affine transformations of a convex polytope result in a convex polytope [5]. This observation, together with Theorem 1 yields the following Theorem:

Theorem 2. *Affine transformations to a convex polytope P that contains the center of a sphere, result in sphere drawings of that polytope's skeletal graph when its vertices are normalized to lie on the sphere, if the origin remains inside P throughout the transformation.*

As we are not assuming that the inscribed polytope obtained from a sphere drawing contains the origin, and we propose to deal with sphere drawings strictly contained in the lower hemisphere, we need an analogous theorem dealing with polytopes not containing the origin.

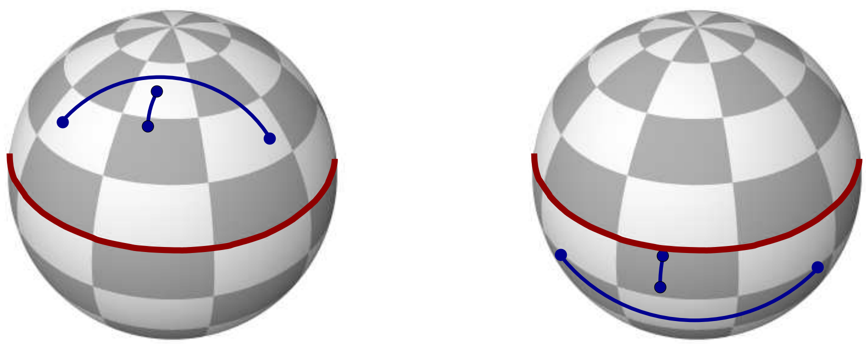
Theorem 3. *A strictly convex polytope P not containing the center of a sphere yields a sphere drawing of that polytope's skeletal graph when its vertices are normalized to lie on the sphere if, for some face f_1, the ray from the origin to any point on the polytope intersects f_1 before any other part of the polytope, and none of the faces of P lie in planes containing the origin.*

Proof Sketch: The face f_1 acts as a shield for rays emanating from the origin. Given a point p of the polytope we can determine its projection p' on the surface of the sphere by taking the intersection of the ray from $(0, 0, 0)$ through p with the sphere. As in Theorem 1, we get an intersection in the spherical drawing if the ray passes through more than one edge of P. However, by the assumption, any such ray must first hit P on f_1. Since P is convex, the ray has exactly one point at which it exits P. If there exists an intersection in the spherical drawing, the ray must intersect edges of P at both its entry and exit points from P. This would contradict the assumption that f_1 is hit first by the ray. □

3.2 Sliding Sphere Drawings to the Equator

The obvious method of "sliding" a sphere drawing down to the lower hemisphere is to do a simple linear scale of the drawing, either by z-coordinates in Euclidean coordinates, or by ϕ in spherical coordinates. This approach, however, does not always work. It is easy to construct an example with two non-intersecting geodesics in the upper hemisphere that must cross on their way to the lower hemisphere if linear scaling is used; see Fig. 2. Therefore, we consider the approach where me manipulate the inscribed polytope.

Theorem 4. *There exists a continuous and crossings-free morph that moves a sphere drawing D, of a maximally planar graph G, to a drawing of G such that the vertices of a chosen face f_0 are on the equator and all others are strictly below the equator, provided that the inscribed polytope P of D is convex.*



(a) (b)

Fig. 2. Linear scaling of the vertices to the southern hemisphere may introduce crossings: (a) the endpoints of the long edge are below those of the short edge; (b) linear scaling could bring all the vertices to the southern hemisphere but at some intermediate stage the two edges intersect.

Proof Sketch: Consider the inscribed convex polytope P corresponding to the sphere drawing D. We have two cases: either P contains the origin or it does not.

Case 1 (P contains the origin): First rotate P so that the outward normal to f_0 is parallel to $(0, 0, 1)$. Let v_0 be the average of the points of f_0. Since P is convex, the segment between the origin and v_0 lies entirely within P. We can thus apply to P a translation along the vector $-v_0$ and be assured that P contains the origin throughout the transformation, so Theorem 1 applies. Now f_0 lies within the xy-plane, so when we project its points onto the sphere, they lie on the equator. Since P is convex, we know all other points of P are on one side of f_0. Since the outward normal of f_0 is pointing up, the other points have to be below f_0, and hence below the equator.

Case 2 (P does not contain the origin): Here we rely on Theorem 3, instead. First we need to show that its precondition is true: that there exists some face f_1 that acts as an shield that eclipses the rest of the polytope from the origin. Since P does not contain the origin, there exists some plane that passes through the origin such that P lies entirely on one side. Thus D has one face, which we conveniently call f_1, that encompasses a half-sphere.

The f_1 face must eclipse the rest of P from the origin. The edges in D that make up f_1 match the edge of the spherical region eclipsed by f_1 in P. Since f_1 is the "outer"-most face, there can be no nodes outside of this region.

We can thus apply any affine transformations to P that maintain f_1's eclipse of the rest of the polytope. We use shearing as it is an affine transformation, straight lines remain straight. If the application of a transformation were to negate f_1's "eclipse" property, then it would have to introduce a clear path from the origin to some edge in P not on f_1.

Shearing and rotation do not affect the origin, so we can apply those while maintaining a valid sphere drawing in the projection. Let v_0 be the centroid of f_1. We rotate P so that v_0 lies in the xy-plane on the line $y = x$. v_0 now lies at $(a, a, 0)$, for some a. Simultaneously shear P in x and y with the factor -1, and v_0 ends up at the origin. We now have a convex polyhedron that contains the origin, and so have reduced the problem to case 1. □

3.3 Sliding Sphere Drawings to the Lower Hemisphere

From Theorem 4 we know that we can transform D into a drawing such that the vertices of a face f_0 are on the equator and all the rest are strictly below the equator. At this stage it is easy to argue that there exists an $\epsilon > 0$ such that we can translate the polytope by an additional ϵ vertically down, so that all the points on the sphere (including those that form f_0) are strictly below the equator.

In practice, however, the valid values of ϵ can be arbitrarily small, making this simple approach unattractive for morphing. The value of ϵ depends on the placement of the vertices of f_0 around the equator. If two vertices of f_0 are near-antipodal, then the edge between them can pass arbitrarily close to the south pole when we translate P strictly below the equator. This would make it difficult to prevent crossings in the spherical drawing. To remedy this problem, we use scaling and shearing (both affine transformations) to the polytope P to make f_0 an equilateral triangle. We consider f_0 by itself in the plane, calculate the transformations necessary to make it equilateral (shear around its centroid until it is isosceles, and then scale to make it equilateral), and apply them to P as a whole.

Our goal is to move all vertices outside of f_0 low enough on the sphere so that we can guarantee f_0 blocks their view of the origin. As we show below, it suffices to move the rest of the points below the Antarctic circle (66^oS, $z \approx -0.9135$) to ensure that they are eclipsed by an f_0 whose vertices lie on the Tropic of Capricorn (23.5^oS, $z \approx -0.3987$). These two values also provide a bound on the area of the straight-line plane drawing obtained as the gnomonic projection of the sphere drawing. With the next theorem we derive the general relation that must exist between these two latitudes in order to guarantee we get a crossing free sphere drawing, as per Theorem 3, and it is straight-forward to verify that that these two values satisfy the relation.

Theorem 5. *There exists a continuous and crossings-free morph that moves a sphere drawing D, of a maximally planar graph G, to a drawing of G such that the all the vertices are strictly below the equator, provided that the inscribed polytope P of D is convex.*

Proof Sketch: Here is the outline of the proof. We begin with f_0 as a triangle in the xy-plane. We apply scaling and shearing to P to transform f_0 into an equilateral triangle. We choose a value z_1 that we want to translate f_0 down to, and calculate a scaling factor s as a function of z_1 and z_3, the highest z-coordinate of any point outside f_0. We scale P in x and y by a factor of $\frac{1}{s}$, and project it back onto the sphere. Note that this leaves f_0 in the xy-plane. The scaling factor was computed so that when we translate P down by z_1 the face f_0 eclipses the rest of P, yielding a valid sphere drawing at each stage by Theorem 3. Since f_0 is now strictly below the equator, and all other nodes are below f_0, the entire drawing is below the equator. Next we provide some of the details about this argument.

We begin where Theorem 4 left off. The inscribed polytope P has the designated face f_0 on the equator and all other vertices in the southern hemisphere. We skip the details about scaling and shearing to P to transform f_0 into an equilateral triangle, and focus on calculating the scaling factor s needed to ensure that when we translate P below the equator, the spherical drawing contains no crossings.

Since we have transformed f_0 into an equilateral triangle, we know exactly where its arcs lie, and can calculate the lowest point on the sphere covered by f_0. We would like to translating P down so that f_0 lies in the plane $z = z_1$ (say, the Tropic of Capricorn). Rotate P so that one of f_0's vertices lies on the y-axis. Then the coordinates of that point are $(0, \sqrt{1-z_1^2}, z_1)$. Since f_0 is equilateral, we can easily find that its other two points are at $(\frac{\sqrt{3}y_1}{2}, \frac{-y_1}{2}, z_1)$ and $(\frac{-\sqrt{3}y_1}{2}, \frac{-y_1}{2}, z_1)$. Since these two are symmetric around the y-axis, we can use the arc between these to find the lowest point of f_0 on the sphere. The midpoint of the spherical arc is the projection of the midpoint of the Euclidean line between these two points, given by the average of the two points:

$$m = (0, \frac{-y_1}{2}, z_1) = (0, \frac{-\sqrt{1-z_1^2}}{2}, z_1)$$

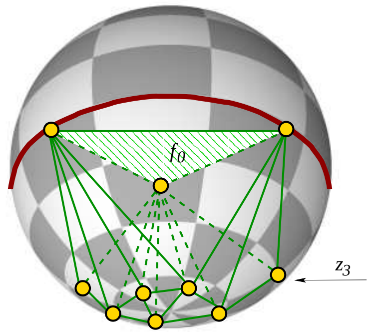
We need its magnitude to project it onto the sphere:

$$||m|| = \sqrt{\frac{-\sqrt{(1-z_1^2)}}{2}^2 + z_1^2} = \sqrt{\frac{1-z_1^2}{4} + z_1^2} = \sqrt{\frac{1}{4} + \frac{3}{4}z_1^2} = \frac{1}{2}\sqrt{3z_1^2+1}$$

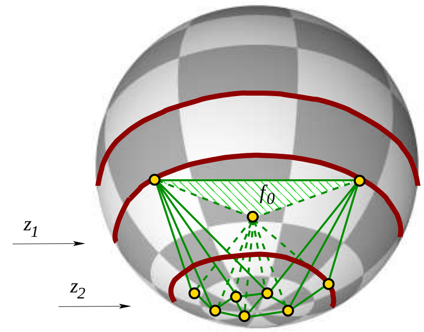
The midpoint m had a z-coordinate of z_1 and so, when projected onto the sphere, it has a z-coordinate of $\frac{z_1}{||m||}$. Thus, the lowest point z_2 of f_0 on the sphere would be

$$z_2 = \frac{z_1}{||m||} = \frac{2z_1}{\sqrt{3z_1^2+1}}.$$

If we move all points of D not in f_0 below z_2, then we can translate P down and guarantee that f_0 still eclipses P from the origin, and thus maintain a valid sphere drawing throughout. Using the Tropic of Capricorn for z_1 gives a z_2 that is above the Arctic Circle, so using the two familiar latitudes guarantees valid sphere drawings throughout. To make sure all vertices outside f_0 are below z_2, we scale P down around the z-axis by some constant factor s. This scaling has



(a)



(b)

Fig. 3. The polytope P has face f_0 on the equatorial plane. The highest z-coordinate of a vertex not on f_0 is given by z_3. We would like to translate the polytope straight down so that f_0 is on the Tropic of Capricorn plane given by $z = z_1$. We ensure that all vertices other than those in f_0 are below the Antarctic circle given by plane $z = z_2$.

the effect of moving all the vertices not in f_0 towards the south pole. We can calculate the scale-factor s necessary to move all nodes below z_2 as follows.

Let z_3 be the maximum z-coordinate of any node in D not in f_0. We would like to scale the point (x, y, z_3) to $(\frac{x}{s}, \frac{y}{s}, z_3)$, such that when it is projected back onto the sphere, its z-coordinate is below z_2. To project $(\frac{x}{s}, \frac{y}{s}, z_3)$ onto the sphere we first find its magnitude. Since the original point lies on the sphere, we have $x^2 + y^2 = 1 - z_3^2$ and the magnitude is given by:

$$\sqrt{\frac{x^2}{s^2} + \frac{y^2}{s^2} + z_3^2} = \sqrt{\frac{x^2 + y^2}{s^2} + z_3^2} = \sqrt{\frac{1 - z_3^2}{s^2} + z_3^2}.$$

As our goal is to have the scaled, projected point lie below z_2, so we need to find a value for s such that: $\frac{z_3}{\sqrt{\frac{1-z_3^2}{s^2} + z_3^2}} < z_2$. Solving for s gives us:

$$s > \sqrt{\frac{1 - z_3^2}{\frac{z_2^2}{z_3^2} - z_3^2}}.$$

Using the scaling factor guarantees all points outside f_0 fall below f_0's arcs on the sphere when projected, and thus f_0 eclipses P throughout the translation, and we can move f_0 on the sphere down to the plane $z = z_1$ with the translation $(0, 0, -z)$.

□

3.4 The Complete Morph

We have shown that we can morph a sphere drawing to another sphere drawing that is entirely in one hemisphere. Then, starting with the source drawing D_s we can morph it to a drawing D'_s that is strictly below the equator. We can do the same with the target sphere drawing D_t and morph it to a sphere drawing D'_t that is strictly below the equator. We then obtain the gnomonic projections D''_s and D''_t of the two drawings onto the plane tangent to the south pole. We then apply the planar morph algorithm to morph between these two plane drawings. Throughout the planar morph, the sphere drawing is the inverse gnomonic projection of the current state of the plane drawing. Finally, we invert the $D_t \to D'_t$ morph to arrive at the target drawing.

In order to perform the planar morph, we must ensure that the outer face in D''_s and D''_t is the same. We must match the upper faces in D'_s and D'_t. Theorem 4 allows us to use whichever face we wish, so this is not a problem.

4 Conclusions and Open Problems

We have shown that under certain conditions we can morph between spherical drawings such that the morph is continuous and intersection-free. There are several open problems:

1. Does there exist a continuous and intersection-free morph between any pair of sphere drawings of an underlying 3-connected graph?
2. In the planar morph stage what is actually computed is not the trajectories of the vertices, but their locations at any stage in the morph. Is there a morph with trajectories of polynomial complexity?
3. One can imagine a morph in which the entire drawing is not transferred to the lower hemisphere would be more visually appealing. Is it possible to apply this morph while keeping the visual representation of the drawings in the upper hemisphere? In graphics, transformations are stacked with other transformations and then inverted (e.g., to rotate around a vector in R^3, one can rotate the whole coordinate system to the x-axis, rotate everything around the x-axis, since that is easy to do, and then apply the inverse of the original rotation). Can we interpolate between the inversions of the two slides so that the morph can use the whole sphere?
4. Is there a more direct way to use spherical barycentric coordinates with interpolating between convex representations of graph to obtain a spherical morph, that doesn't involve reducing the problem to a planar morph?

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References

1. P. Alfeld, M. Neamtu, and L. L. Schumaker. Bernstein-Bézier polynomials on circle, spheres, and sphere-like surfaces. *Computer Aided Geometric Design Journal*, 13:333–349, 1996.
2. B. Aronov, R. Seidel, and D. Souvaine. On compatible triangulations of simple polygons. *CGTA: Computational Geometry: Theory and Applications*, 3:27–35, 1993.
3. T. Beier and S. Neely. Feature-based image metamorphosis. In E. E. Catmull, editor, *SIGGRAPH '92*, number 2, pages 35–42, July 1992.
4. S. S. Cairns. Deformations of plane rectilinear complexes. *American Math. Monthly*, 51:247–252, 1944.
5. H. Coxeter. *Introduction to Geometry*. Wiley, 1961.
6. C. Erten, S. G. Kobourov, and C. Pitta. Intersection-free morphing of planar graphs. In *11th Symposium on Graph Drawing*, pages 320–331, 2003.
7. M. Floater and C. Gotsman. How to morph tilings injectively. *Journal of Computational and Applied Mathematics*, 101:117–129, 1999.
8. E. Goldstein and C. Gotsman. Polygon morphing using a multiresolution representation. In *Graphics Interface '95*, pages 247–254, 1995.
9. J. Gomes, L. Darsa, B. Costa, and D. M. Vello. *Warping and Morphing of Graphical Objects*. Morgan Kaufmann, 1999.
10. C. Gotsman, X. Gu, and A. Sheffer. Fundamentals of spherical parameterization for 3D meshes. In *SIGGRAPH '03*, pages 358–363, 2003.
11. C. Gotsman and V. Surazhsky. Guaranteed intersection-free polygon morphing. *Computers and Graphics*, 25(1):67–75, Feb. 2001.
12. T. He, S. Wang, and A. Kaufman. Wavelet-based volume morphing. In *Proceedings of the Conference on Visualization*, pages 85–92, 1994.
13. J. F. Hughes. Scheduled Fourier volume morphing. *Computer Graphics*, 26(2):43–46, July 1992.
14. T. Samoilov and G. Elber. Self-intersection elimination in metamorphosis of two-dimensional curves. *The Visual Computer*, 14:415–428, 1998.
15. T. W. Sederberg and E. Greenwood. A physically based approach to 2-D shape blending. In *SIGGRAPH*, pages 25–34, July 1992.
16. C. Thomassen. Deformations of plane graphs. *J. Combin. Theory Ser. B*, 34:244–257, 1983.
17. W. T. Tutte. How to draw a graph. *Proc. London Math. Society*, 13(52):743–768, 1963.